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On Product Integration of Certain Generalized Operator-Valued Functions

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A product integral solution of the abstract evolution equation $u'(t) = H_I(t)\, u(t)$ is obtained in the event that $H_I(t)$ belongs to a class of generalized operator-valued functions involving singular perturbations in the interaction representation ($H_I(t) = ie^{-itH_0} V e^{itH_0}$, where H_0 is a power of $(-\Delta)^{1/2}$ in $L^2(R^n)$, and V is a suitable distribution).

For nearly a century, product integration has been an effective tool in the solution of linear evolution equations:

$$u'(s) = A(s)\, u(s).$$

The general procedure, first developed by Volterra to deal with matrix-valued functions $A(s)$, was subsequently expanded by several authors to the case in which $A(s)$ is bounded operator-valued. In [7], Kato showed that under certain circumstances, $A(s)$ could even be unbounded for each s. The purpose of this note is to demonstrate that for a wide class of examples, an interpretation of the product integral is feasible in the event that $A(s)$ is for each s not even an operator, but merely a quadratic form. Our approach is motivated by Segal's treatment of singular perturbations in [9, 10], the results in [4, 5], and the procedure of Dollard and Friedman (cf. [2, 3]) for product integrating unbounded operator-valued functions.

In [4, 5], the time-dependent Schrödinger equation $u'(t) = iH_I(t)\, u(t)$, where $H_I(t)$ is a generalized operator-valued function, was studied. On a suitable domain in the underlying Hilbert space, $H_I(t)\,(\phi, \psi)$ is assumed to be a sesquilinear form for each t, and is not necessarily closed nor bounded below. Nevertheless, in the event $H_I(s) = e^{-isH_0} V e^{isH_0}$, where $H_0 = ((1/i)(d/dx))^m$ in $L^2(R^1)$, with $m \geqslant 2$, or $H_0 = (-\Delta)^{m/2}$ in $L^2(R^n)$, $n \geqslant 2$, and $m > 2n - 1$, and V is a distribution belonging to a suitable class, a "unitary propagator" $U(t, s)$ is obtained, as a time-ordered exponential series con-

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verging in the uniform operator topology. The key to the success of this approach is L^2-boundedness of the formal expression $\int_0^t e^{-isH_0} V e^{isH_0}\, ds$ which, under Fourier transformation, becomes a bounded integral operator on R^n.

In view of the close relationship between time-ordered exponential series and product integrals (for example, the existence of the former is used to define the latter for $A(\cdot) \in L_s^1[a, b]$ in [2, 3]), it is interesting to ask whether an alternative development using product integrals is feasible in the context of [4, 5]. Our approach uses the established existence of $U(t, s)$, and a modification of its continuity as a function of V (cf. [4, Theorem 4.2; 5, Theorem 2.10]), as does that of Dollard and Friedman in their proof of the existence of product integrals for certain unbounded operator-valued functions [2, Theorem 12]. More precisely, let $\{V_n\}$ be a sequence of real-valued functions belonging to the Schwartz class of rapidly decreasing functions on the line, and whose Fourier transforms converge pointwise a.e. to $\hat{V}$ which, for simplicity, we now assume belongs to $L^\infty(R^1)$ (we are in the setting of [4]). Then the product integrals of the bounded, self-adjoint operator-valued functions $H_{I_n}(s) = e^{-isH_0} V_n e^{isH_0}$, where $H_0 = ((1/i)(d/dx))^m$, $m \geqslant 2$, exist in the strong operator topology, and coincide with the time-ordered exponential series $U_{V_n}(t, s)$ (cf. [3, Sect. 3.5, Corollary 2]). Due to the continuity of U_V as a function of V, $U_V(t, s) = \text{s-lim}_{n \to \infty} U_{V_n}(t, s)$, so we obtain

$$U_V(t, s) = \operatorname*{s-lim}_{n \to \infty} \prod_s^t e^{H_{I_n}(s)\, ds}, \tag{1}$$

which is analogous to the resulting product integral in [2, Theorem 12].

We use this phenomenon (1) as a starting point for a general treatment of product integration of generalized functions, by demonstrating a natural means of selecting the approximating family $\{V_n\}$ that is intrinsic to the setting under discussion and exploits the boundedness of the first-order terms in the time-ordered series. This approximation, similar to that used by Chernoff [1] in establishing product formulae for strongly continuous semigroups generated by Friedrichs extensions, requires that we first transform the setting in [4, 5] to the language of quadratic forms.

Let $H_0 = ((1/i)(d/dx))^m$ in $L^2(R^1)$, or $H_0 = (-\Delta)^{m/2}$ in $L^2(R^n)$ if $n \geqslant 2$, with m as described above. Set

$$\mathscr{D}^\alpha(R^n) = \{\hat{V} \mid \hat{V} \text{ is measurable on } R^n, \text{ bounded on bounded subsets,}$$
$$\hat{V}(-k) = \overline{\hat{V}(k)}, \text{ and } \hat{V}(k) = O(|k|^\alpha) \text{ as } |k| \to \infty\}.$$

For $\hat{V} \in \mathscr{D}^\alpha(R^n)$, let

$$D(V) = \{\phi \in L^2(R^n) \mid \langle |\hat{V}| * |\hat{\phi}|, |\hat{\phi}| \rangle < \infty\},$$

and for $\phi \in D(V)$, define

$$V(\phi, \phi) = \langle \hat{V}^* \hat{\phi}, \hat{\phi} \rangle.$$

Let $Q(H_0) \equiv H^1$ denote the form domain of H_0, and let $H^1 \subset H \subset H^{-1}$ be the scale of spaces associated with $|H_0|$. In light of the Proposition in [6], $V \in L(H^1, H^{-1})$, provided $\alpha = (m-(1+\varepsilon))/2$ in $L^2(R^1)$, and $\alpha = (m+1-\varepsilon)/2 - n$ in $L^2(R^n)$, $n \geqslant 2$.

Now recall that the formal expression $\int_s^t e^{-iuH_0} V e^{iuH_0}\, du$ gives rise, under Fourier transformation, to a bounded self-adjoint integral operator with kernel

$$K(x, y) = \frac{e^{-it(|x|^m - |y|^m)} - e^{-is(|x|^m - |y|^m)}}{i(|x|^m - |y|^m)} \hat{V}(x-y)$$

(the absolute value may be omitted in $L^2(R^1)$), and $\|\int_s^t e^{-iuH_0} V e^{iuH_0}\, du\| \leqslant \sup_{x \in R^n} \int_{R^n} |K(x, y)|\, dy$ (cf. [5, Theorem 2.1]).

Following [3, Sect. 3.3, Definition 3.4], we define the mean-value approximant corresponding to a fixed partition of $[s, t]$:

DEFINITION. Let $s = s_0 < s_1 < \cdots < s_n = t$. For $k = 0, 1, \ldots, n-1$, set

$$\bar{M}_{\pi_k} = \frac{1}{\Delta s_k} H[s_{k-1}, s_k] = \frac{1}{\Delta s_k} \int_{s_{k-1}}^{s_k} e^{-iuH_0} V e^{iuH_0}\, du.$$

The *mean-value approximant* associated to π is the step function whose value on $(s_{k-1}, s_k]$ is $\bar{M}_{\pi_k}$.

Remark. Whereas the mean-value approximant in [3] is defined in the case of integrands that are bounded operator-valued functions, the situation considered here is much more pathological. Nevertheless, $M_\pi(\cdot)$ still exists as a bounded self-adjoint operator-valued step function. Following [3], we define its product integral: for a fixed partition π, set

$$\begin{aligned} E_\pi(u, s) &= e^{i(u-s_0)\bar{M}_{\pi_1}}, && u \in [s_0, s_1), \\ &= e^{i(u-s_1)\bar{M}_{\pi_2}} e^{iH[s_1, s_0]}, && u \in [s_1, s_2), \\ &\;\vdots \\ &= e^{i(u-s_{n-1})\bar{M}_{\pi_n}} e^{iH[s_{n-1}, s_{n-2}]} \cdots e^{iH[s_1, s_0]}, && u \in [s_{n-1}, s_n]. \end{aligned}$$

Clearly $E_\pi(u, s)$ is unitary for each $u \in [s, t]$, and is continuous in the uniform operator topology as a function of u (it was shown in [4, Corollary 2.4; 5, Proposition 2.2] that $H[u, s]$ is a continuous function of u in the uniform operator topology). Moreover, the function $M_\pi(u)\, E_\pi(u)$ is integrable on $[s, t]$ and the derivative (in the uniform topology) of $E_\pi(u, s)$ is $iM_\pi(u)\, E_\pi(u, s)$, except at division points of the partition. Hence

$$E_\pi(t, s) = I + i \int_s^t M_\pi(u)\, E_\pi(u, s)\, du$$

follows in much the same way as the standard theory in [3].

We now introduce our canonical approximating family. For fixed $\varepsilon > 0$, let

$$H_{I,\varepsilon}(u) = e^{-iuH_0}(I + \varepsilon |H_0|)^{-1}\, V e^{iuH_0}.$$

Then $H_{I,\varepsilon}(u) \in L(H^1)$, for $(I + \varepsilon |H_0|)^{-1} \in L(H^{-1}, H^1)$, and $H_{I,\varepsilon}(u)$ is strongly continuous in $L(H^1)$. In addition, we see by taking Fourier transforms that $\int_s^t H_{I,\varepsilon}(u)\, du$ corresponds to an integral operator with kernel

$$K_\varepsilon(x, y) = \frac{e^{-it(|x|^m - |y|^m)} - e^{-is(|x|^m - |y|^m)}}{i(|x|^m - |y|^m)} \frac{\hat{V}(x - y)}{1 + \varepsilon |x|^m}$$

(the absolute values in the first factor may be omitted in $L^2(R^1)$). Although [5, Theorem 2.1] is not directly applicable, it is easy to see directly that $\int_s^t H_{I,\varepsilon}(u)\, du \in L(H)$ for all $s, t \in R^1$, as $\sup_{x \in R^1} \int |K_\varepsilon(x, y)|\, dy < \infty$. Moreover, since $1/(1 + \varepsilon |x|^m) \leqslant 1$ for all $\varepsilon > 0$, $x \in R^1$, we may essentially ignore this factor in the estimates needed to establish boundedness of the higher-order terms in the corresponding time-ordered series $U_\varepsilon(t, s)$. Indeed, a straightforward check reveals that these terms are all bounded on $H = L^2(R^n)$, and that the series $U_\varepsilon(t, s)$ converges in the uniform operator topology of $L(H)$.

Now, working in the Banach space $L(H^1)$, we define, for a fixed partition π, the mean-value approximant $M_{\pi_\varepsilon}(\cdot)$: that is, for $u \in (s_{k-1}, s_k]$,

$$M_{\pi_\varepsilon}(u) = \frac{1}{\Delta s_k} \int_{s_{k-1}}^{s_k} e^{isH_0}(I + \varepsilon |H_0|)^{-1}\, V e^{isH_0}\, ds.$$

By virtue of the Dollard–Friedman theory of *strong* product integration (cf. [3, Sect. 3.5, Theorem 5.1]), we have that

$$U_\varepsilon(t, s) \equiv \operatorname*{s\text{-}lim}_{\mu(\pi) \to 0} E_{\pi,\varepsilon}(t, s) \qquad \text{exists in } L(H^1),$$

and coincides with the time-ordered exponential series, which we also denote $U_\varepsilon(t, s)$. In particular, for all $\phi \in H^1$,

$$U_\varepsilon(t, s)\, \phi = \lim_{\mu(\pi) \to 0} E_{\pi,\varepsilon}(t, s)\, \phi \qquad \text{in } H^1, \text{ hence in } H.$$

Using the facts that all operators involved are unitary in H, and H^1 is dense in H, we obtain

$$U_\varepsilon(t, s) = \underset{\mu(\pi)\to 0}{\text{s-lim}}\ E_{\pi,\varepsilon}(t, s) \qquad \text{in } L(H). \tag{2}$$

For each $\varepsilon > 0$,

$$\left|\frac{1}{1+\varepsilon|x|^m}\hat{V}(x-y)\right| \leqslant |\hat{V}(x-y)| \quad \text{and} \quad \frac{1}{1+\varepsilon|x|^m}\hat{V}(x-y) \to \hat{V}(x-y)$$

a.e. as $\varepsilon \to 0^+$. By the proof of the established continuity of the unitary propagator U (cf. [5, Theorem 2.10]), we obtain

$$U_V(t, s) = \lim_{\varepsilon\to 0^+} U_\varepsilon(t, s) \tag{3}$$

in $L(H)$. Combining (2) and (3), we obtain

$$U_V(t, s) = \underset{\varepsilon\to 0^+}{\text{s-lim}}\ \lim_{\mu(\pi)\to 0} E_{\pi,\varepsilon}(t, s) \qquad \text{in } L(H). \tag{4}$$

Using more transparent notation, we may restate (4) as our main result:

THEOREM. *For $V \in \mathscr{D}^\alpha(R^n)$, and $t, s \in R^1$,*

$$U_V(t, s) = \underset{\varepsilon\to 0^+}{\text{s-lim}}\ \lim_{n\to\infty} \prod_{k=1}^{n} \exp^{(i\int_{s_{k-1}}^{s_k} e^{-iuH_0(I+\varepsilon|H_0|)^{-1}} V e^{iuH_0}\,du)}. \tag{5}$$

Remark. Equation (5) demonstrates how close we are to a bona fide product integral, for if we could interchange the limiting procedures, we would obtain

$$U_V(t, s) = \underset{n\to\infty}{\text{s-lim}} \prod_{k=1}^{n} \exp^{(i\int_{s_{k-1}}^{s_k} e^{-iuH_0} V e^{iuH_0}\,du)}.$$

This interchange, however, appears to be quite difficult, and is undoubtedly related to the problem of establishing the Trotter product formula for unitary groups generated by form sums of positive self-adjoint operators (cf. [1, p. 115; 8]). That our product integral satisfies the requisite properties is verified in [5, 6].

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